Interest Rate Convexity and the Volatility Smile*

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Abstract

When pricing the convexity effect in irregular interest rate derivatives such as, e.g., Libor-in-arrears or CMS, one often ignores the volatility smile, which is quite pronounced in the interest rate options market. This note solves the problem of convexity by replicating the irregular interest flow or option with liquidly traded options with different strikes thereby taking into account the volatility smile. This idea is known among practitioners for pricing CMS caps. We approach the problem on a more general scale and apply the result to various examples.

Key words: interest rate options, volatility smile, convexity, option replication

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Contents

1 Introduction

Pricing irregular interest cash flows such as Libor-in-arrears or CMS requires a convexity correction on the corresponding forward rate. This convexity correction involves the volatility of the underlying rate as traded in the cap/floor or swaption market. In the same spirit, an option on an irregular rate, such as an in-arrears cap or a CMS cap, is often valued by applying a Black & Scholes model with a convexity adjusted forward rate and a convexity adjusted volatility, see e.g. [1], [3], [5], [6]. However, to a large extent, this approach ignores the volatility smile, which is quite pronounced in the cap/floor market. This note solves this problem by replicating the irregular interest flow or option with liquidly traded options with different strikes. This approach is well-known among practitioners for pricing CMS caps. There is a certain overlap of the present paper with a recent paper by Hagan, [4]. However, we approach the problem from a different and more generic point of view and apply the result to various examples related to interest rates.

We illustrate the approach by numerical examples based on market data for the volatility smile in the interest rate derivatives market. Comparing the results of the "smile convexity" with the results from a more simplistic approach based solely on adjusted forward rates (which is still frequently used in practice) we show that the smile effect on the convexity is by no means just an additional tiny quantity and cannot be ignored in practice.

Another important consequence of the replication approach is that one obtains immediately the respective simultaneous delta and vega hedges in terms of liquidly traded options.

2 Setup of the problem

Consider a financial underlying with price $Y > 0$ at time T. Suppose there is a liquid market for plain vanilla options on this underlying with all possible strikes K. Somewhat more general, we suppose that for all $K \geq 0$ the price $P(K)$ today of the "plain vanilla" derivative with payoff

$$g(Y)\max(Y-K,0)$$

at time T is known. Here g is a certain function with $g(y) > 0$ whose role will become clear in the examples below. In the simplest case $g \equiv 1$ and the liquidly traded options are just calls on the underlying Y.

Our goal is to price an exotic contingent claim with payoff

$$f(Y),$$

at time T, where $f : (0, \infty) \to (-\infty, \infty)$.

The idea is to replicate the exotic payoff $f(Y)$ by a portfolio of traded derivatives $g(Y)\max(Y - K, 0)$ with different strikes K. If this is possible, then the replication is the key to incorporating the volatility smile of the liquid options into the pricing of the exotic derivative.

So we are looking for a representation

$$f(Y) = C + \int_{[0,\infty)} g(Y)\max(Y - K, 0)\mathrm{d}\mu(K) \tag{1}$$

with some locally finite signed measure μ on $[0, \infty)$ and some constant C. If such a representation exists, then by no-arbitrage the fair price $P(f(Y))$ of the contingent claim $f(Y)$ is given by

$$\boxed{P(f(Y)) = C \cdot B(0, T) + \int_{[0,\infty)} P(K)\mathrm{d}\mu(K),} \tag{2}$$

where $B(0, T)$ denotes the price of a zero bond with maturity T, and provided the integral on the right hand side is well-defined.

Remark. The importance of the formula (2) goes beyond the issue of just pricing a derivative with payoff $f(Y)$, since, at the same time, it provides us with an explicit strategy for a simultaneous delta and vega hedge of the derivative $f(Y)$ in terms of liquidly traded products $g(Y)\max(Y - K, 0)$. In practice one would discretize the integral appropriately to get an (approximate) hedging strategy in a finite set of products $g(Y)\max(Y - K_i, 0)$ with different strikes $K_i, i = 1, \ldots$

3 Target applications related to irregular interest rate derivatives

The examples that we have in mind and that we are trying to approach with the above idea are irregular interest rate cash flows or options on such rates. We denote by $B(t, T)$ the price at time t of a zero bond with maturity T.

Example 1:
Let $Y = L(T_1, T_2)$ be the money market rate (Libor) for the interest period $[T_1, T_2]$,

$$L(T_1, T_2) = \frac{\frac{1}{B(T_1,T_2)} - 1}{\Delta},$$

with period length Δ according to the given day count convention. In the interest derivatives market, caplets, i.e., call options on this underlying with

payoff $\Delta \max(L(T_1, T_2) - K, 0)$, are quite liquid for (more or less) any strike rate K. By default, the caplet pays in arrears, i.e., at time T_2. If we are looking for an "exotic" derivative such as, for example, Libor-in-arrears, which is the payment of

$$\Delta\, L(T_1, T_2) \text{ at time } T_1,$$

or an in-arrears caplet,

$$\Delta \max(L(T_1, T_2) - \bar{K}, 0) \text{ paid at time } T_1,$$

we have to take care of convexity effects caused by the fact that the payment occurs at the non-standard point in time T_1. These effects are well-known and there exist valuation formulae that yield exact or sufficiently precise results in the framework of the standard lognormal interest rate model, see e.g. [3], [1], [6]. However these approaches do not take into account the volatility smile in the cap market.

The standard caplet paying at time T_2 is equivalent to a product that pays the amount

$$\frac{\Delta}{1 + \Delta L(T_1, T_2)} \max(L(T_1, T_2) - K, 0)$$

at time T_1. So the market gives us prices for the time T_1 payout

$$g(L(T_1, T_2)) \max(L(T_1, T_2) - K, 0) \text{ where } g(y) = \frac{\Delta}{1 + \Delta y}$$

for any strike K. Our goal is to price an "exotic" payoff

$$f(L(T_1, T_2))$$

at time T_1, where, for example, $f(y) = \Delta\, y$ in case of a Libor-in-arrears, or, $f(y) = \Delta \max(y - \bar{K}, 0)$ in the case of an in-arrears caplet.

Assuming a linear Libor rate model as in [1], Section 3.1, we can even handle the somewhat more general case of the payment of $f(L(T_1, T_2))$ at some arbitrary time $p \geq T_1$. In the linear rate model it is assumed that

$$\frac{B(T_1, p)}{B(T_1, T_2)} = 1 + \beta_p L(T_1, T_2),\ p \geq T_1,$$

where

$$\beta_p = \frac{\frac{B(0,p)}{B(0,T_2)} - 1}{L_0(T_1, T_2)} \tag{3}$$

and where the forward rate $L_0(T_1, T_2)$ is given by

$$L_0(T_1, T_2) = \frac{\frac{B(0,T_1)}{B(0,T_2)} - 1}{\Delta}.$$

In case of $p = T_1$ we get $\beta_p = \Delta$ and the assumption of the linear rate model is no restriction in this case.

Then, under the assumption of the linear rate model the payment of $f(L(T_1, T_2))$ at time p is equivalent to the payment of

$$\frac{1 + \beta_p L(T_1, T_2)}{1 + \Delta L(T_1, T_2)} f\left(L(T_1, T_2)\right) \tag{4}$$

at time T_1,

The next two examples are related to interest rate derivatives whose payoff is linked to a swap rate, so-called CMS[1] products.

Let $Y = C(T_0, T_n)$ denote the fair swap rate at time T_0 for a swap with reference dates $T_0 < T_1 < \cdots < T_n$,

$$C(T_0, T_n) = \frac{1 - B(T_0, T_n)}{\sum_{i=1}^{n} \Delta_i B(T_0, T_i)},$$

with Δ_i denoting the length of the interval $[T_{i-1}, T_i]$.

A swaption is an option to enter at time T_0 into a swap with given fixed rate K. The payoff of a *cash-settled payer swaption* at time T_0 is

$$\left(\sum_{i=1}^{n} \frac{\Delta_i}{(1 + C(T_0, T_n))^{T_i - T_0}} \right) \max(C(T_0, T_n) - K, 0). \tag{5}$$

For a *physically-settled payer swaption* the payoff at time T_0 is

$$\left(\sum_{i=1}^{n} \Delta_i B(T_0, T_i) \right) \max(C(T_0, T_n) - K, 0). \tag{6}$$

The sum in front of the max in both payoffs (5), (6) is the so-called present (or dollar) value of one basis point factor, DV01. For a cash-settled swaption, by convention, this present value factor is calculated as if the yield curve was flat. Swaptions are quite liquid for “all” strikes. The market does not make a significant difference in pricing a cash- or physically-settled swaption, and both are priced under a lognormal Black model.

[1]Constant maturity swap.

Example 2:
The payoff of a cash-settled swaption is again of the form $g(C(T_0, T_n)) \max(C(T_0, T_n) - K, 0)$ where

$$g(y) = \mathrm{DV01}(y) = \sum_{i=1}^{n} \frac{\Delta_i}{(1+y)^{T_i - T_0}}. \tag{7}$$

The exotic payoffs $f(C(T_0, T_n))$ of interest are, for example, the payment of

$$f(C(T_0, T_n)) = C(T_0, T_n), \tag{8}$$

paid at time T_0. This is like a cash flow in a CMS swap, but paid in advance. Another example of interest is a CMS caplet,

$$f(C(T_0, T_n)) = \max(C(T_0, T_n) - \bar{K}, 0) \tag{9}$$

to be paid at time T_0.

In case the payoff is scheduled for time $p \geq T_0$, we can approximately think of a derivative with payoff

$$f(C(T_0, T_n)) = \frac{C(T_0, T_n)}{(1 + C(T_0, T_n))^{p - T_0}} \text{ for a CMS swap} \tag{10}$$

$$f(C(T_0, T_n)) = \frac{\max(C(T_0, T_n) - \bar{K}, 0)}{(1 + C(T_0, T_n))^{p - T_0}} \text{ for a CMS caplet.} \tag{11}$$

Example 3:
This example is closely related to Example 2 but gives a more general result under the additional assumption of a linear swap rate model as investigated in [5], [6], [1]. Also the replication argument is no longer based on cash-settled swaptions but on physically-settled swaptions. The market prices swaptions based on a lognormal model for the swap rate $C(T_0, T_n)$ under the so-called swap measure $\mathbf{Q}_{\text{Swap}}$, which is the equivalent martingale measure referring to the numeraire

$$N_t = \sum_{i=1}^{n} \Delta_i B(t, T_i).$$

The payoff of a standard swaption at time T_0 is

$$\sum_{i=1}^{n} \Delta_i B(T_0, T_i) \max(C(T_0, T_n) - K, 0) = N_{T_0} \max(C(T_0, T_n) - K, 0),$$

and its price is given by

$$P(K) = N_0 \, \mathbb{E}_{\mathbf{Q}_{\text{Swap}}} \max(C(T_0, T_n) - K, 0). \tag{12}$$

An exotic derivative paying $f(C(T_0, T_n))$ at some time $p \geq T_0$ is equivalent to a product that pays $B(T_0, p)f(C(T_0, T_n))$ at time T_0. Under the assumption of a linear swap rate model (see e.g. [5], [1], Section 3.2) we have

$$\frac{B(T_0, p)}{N_{T_0}} = \alpha + \beta_p C(T_0, T_n),$$

and the price of $f(C(T_0, T_n))$ to be paid at time p is given by

$$N_0 \, \mathbb{E}_{\mathbf{Q}^{\mathrm{Swap}}}(\alpha + \beta_p C(T_0, T_n)) f(C(T_0, T_n)). \tag{13}$$

Here α and β are defined by

$$\alpha = \frac{1}{\sum_{i=1}^{n} \Delta_i}, \tag{14}$$

$$\beta_p = \frac{\frac{B(0,p)}{\sum_{i=1}^{n} \Delta_i B(0,T_i)} - \alpha}{C_0(T_0, T_n)}, \tag{15}$$

with forward swap rate $C_0(T_0, T_n)$ given by

$$C_0(T_0, T_n) = \frac{B(0, T_0) - B(0, T_n)}{\sum_{i=1}^{n} \Delta_i B(0, T_i)}. \tag{16}$$

In view of equations (12) and (13), in our replication approach we are therefore looking for a representation (1) of the form

$$(\alpha + \beta_p Y) f(Y) = C + \int_{[0,\infty)} \max(Y - K, 0) \mathrm{d}\mu(K). \tag{17}$$

4 The result

Proposition 1 *The exotic payoff $f(Y)$ allows for a replication (1) with some locally finite signed measure μ on $[0, \infty)$ and some constant C if and only if*
(i) $\lim_{x \downarrow 0} f(x) = C$, and,
(ii) the function $\frac{f-C}{g}$, extended to the domain of definition $[0, \infty)$ by setting $\frac{f(0)-C}{g(0)} = 0$, is a difference of convex functions on $[0, \infty)$.
The measure μ is then generated by the following right continuous generalized distribution function (function of locally bounded variation)

$$\mathrm{d}\mu(y) = \mathrm{d}D_+ \left(\frac{f(y) - C}{g(y)} \right),$$

with D_+ denoting the right hand derivative and defining $D_+ \left(\frac{f-C}{g} \right)(0-) = 0$.

Proof. We identify the signed measure μ with its generalized right continuous distribution function, $\mu(0-) = 0$. Integrating by parts the representation (1) is equivalent to

$$\begin{aligned}
\frac{f(y) - C}{g(y)} &= \int_{[0,\infty)} \max(y - K, 0)\mathrm{d}\mu(K) \\
&= \int_{[0,y]} (y - K)\mathrm{d}\mu(K) \\
&= y[\mu(y) - \mu(0-)] - \int_{(0,y]} K\mathrm{d}\mu(K) \\
&= y\ \mu(y) - y\ \mu(y) + \int_0^y \mu(K)\mathrm{d}K \\
&= \int_0^y \mu(K)\mathrm{d}K.
\end{aligned}$$

The assertion now follows. □

Remark. It is well known that from a universe of prices of call options[2],

$$P(K) = \mathbb{E}\max(Y - K, 0),$$

for all strikes $K > 0$ one can extract the implied risk-neutral distribution $F_Y(x) = \mathbf{P}(Y \leq x)$ of the underlying asset Y at maturity by taking the derivative of the call prices w.r.t. the strike,

$$F_Y(x) = \mathbf{P}(Y \leq x) = 1 + D_+P(x),$$

see e.g. [2]. Then, by no-arbitrage the price of any hedgeable contingent claim can be calculated as the expectation w.r.t. this distribution. In particular, for a derivative with payoff $f(Y)$ the price is

$$P(f(Y)) = \mathbb{E}(f(Y)) = \int_0^\infty f(x)\mathrm{d}F_Y(x) = \int_0^\infty f(x)\mathrm{d}D_+P(x).$$

We will now analyze how this well-known result relates to our replication idea. Assume that the payoff function f is a difference of convex functions,

$$f(x) = C + \int_0^x \mu(z)\mathrm{d}z,$$

where μ is a function of locally bounded variation on $[0, \infty)$. Then under appropriate integrability conditions on f by applying integration by parts

[2] We assume for simplicity that risk-less interest rates are zero.

twice one gets

$$P(f(Y)) = \int_0^\infty f(x)\mathrm{d}D_+P(x) = C + \int_{[0,\infty)} P(K)\mathrm{d}\mu(K),$$

clarifying the link between the two approaches.

5 Application to the examples

Let us apply the above result to our examples. As we shall see in these examples the measure μ normally possesses a jump and the constant C often vanishes.

Example 1:
We investigate the general case of a payoff $f(L(T_1, T_2))$ at time $p \geq T_1$ and assume a linear rate model, which is no restriction in case that $p = T_1$.

For the Libor payoff $f(L(T_1, T_2)) = \Delta L(T_1, T_2)$ paid at some arbitrary time $p \geq T_1$ as in (4) the generalized distribution function μ, which gives us the replication of the price in terms of caplet prices, is obtained from

$$\begin{aligned} \frac{f(y)(1+\beta_p y)}{g(y)(1+\Delta y)} &= y(1+\beta_p y) \\ \mu(y) &= 1 + 2\beta_p y, \;\; y \geq 0, \\ \mu(0-) &= 0, \end{aligned}$$

with β_p given by (3). This yields the following formula for the price of a Libor expressed in terms of caplet prices $P(K)$ with different strikes K,

$$\begin{aligned} &P(\Delta L(T_1, T_2) \text{ paid at time } p) \\ &= P(0) + \int_0^\infty P(K) 2\beta_p \mathrm{d}K. \end{aligned} \tag{18}$$

Observe that $P(0)/\Delta$ is the forward rate $L_0(T_1, T_2)$ discounted from time T_2 .

For the caplet $f(L(T_1, T_2)) = \Delta \max(L(T_1, T_2) - \bar{K}, 0)$ paid at time $p \geq T_1$ the corresponding distribution function μ is calculated as

$$\frac{f(y)(1+\beta_p y)}{g(y)(1+\Delta y)} = \max(y - \bar{K}, 0)(1+\beta_p y)$$

$$\mu(y) = \begin{cases} 0 & \text{if } y < \bar{K}, \\ 1 - \beta_p \bar{K} + 2\beta_p y & \text{if } y \geq \bar{K}, \end{cases}$$

and the resulting pricing equation is

$$\begin{aligned} &P(\Delta \max(L(T_1,T_2)-\bar{K},0) \text{ paid at time } p) \\ &\quad = P(\bar{K})(1+\beta_p\bar{K}) + \int_{\bar{K}}^{\infty} P(K) 2\beta_p \mathrm{d}K. \end{aligned} \tag{19}$$

Remark. In case the cap market quotes no smile, the Libor $L(T_1, T_2)$ follows a standard Black model with volatility σ under the time T_2-forward measure, i.e.,

$$\begin{aligned} L(T_1,T_2) &= L_0(T_1,T_2)\exp(\sigma W_{T_1} - \sigma^2 T_1/2) \\ L_0(T_1,T_2) &= \left(\frac{B(0,T_1)}{B(0,T_2)} - 1\right)/\Delta, \end{aligned}$$

with some Wiener process W. In this case, interchanging the order of the $\mathrm{d}K$ integral with the expectation that leads to the prices $P(K)$ in formula (18), the result reduces to the well-known convexity adjustment formula for Libor-in-arrears,

$$\begin{aligned} &P(L(T_1,T_2) \text{ paid at time } T_1) \\ &\quad = B(0,T_1)\, L_0(T_1,T_2)\left(1 + \frac{\Delta L_0(T_1,T_2)(\exp(\sigma^2 T_1)-1)}{1+\Delta L_0(T_1,T_2)}\right), \end{aligned} \tag{20}$$

see, e.g. [1], Formula (14). The expression after the discount factor $B(0, T_1)$ on the right hand side of (20) is the so-called convexity adjusted forward Libor.

Example 2:
Let us start with the payment of a swap rate $C(T_0, T_n)$ at time $p \geq T_0$, see (10). In this case

$$\frac{f(y)}{g(y)} = \frac{y}{\mathrm{DV01}(y)},$$

where

$$\mathrm{DV01}(y) = \sum_{i=1}^{n} \frac{\Delta_i}{(1+y)^{T_i - p}}. \tag{21}$$

The generalized distribution function μ is then

$$\begin{aligned} \mu(y) &= \frac{\mathrm{DV01}(y) - y\,\mathrm{DV01}'(y)}{\mathrm{DV01}^2(y)},\ y \geq 0 \\ \mu(0-) &= 0. \end{aligned}$$

In view of $\mathrm{d}\mu(0) = \frac{1}{\mathrm{DV01}(0)} = \frac{1}{\sum_{i=1}^{n}\Delta_i}$ we obtain the valuation formula

$$P(C(T_0, T_n)) = \frac{1}{\sum_{i=1}^{n} \Delta_i} P(0) + \int_0^\infty P(K) \left(\frac{\text{DV01}(K) - K\,\text{DV01}'(K)}{\text{DV01}^2(K)} \right)' \mathrm{d}K. \quad (22)$$

For the exotic option with payoff given by Equation (11) consisting of a call option on the swap rate $C(T_0, T_n)$, which pays off at time $p \geq T_0$, the replication by plain-vanilla swaptions is derived as

$$\frac{f(y)}{g(y)} = \frac{\max(y - \bar{K}, 0)}{\text{DV01}(y)}$$

$$\mu(y) = \begin{cases} 0 & \text{if } y < \bar{K}, \\ \frac{\text{DV01}(y) - (y - \bar{K})\text{DV01}'(y)}{\text{DV01}^2(y)} & \text{if } y \geq \bar{K}, \end{cases}$$

and we end up with the valuation formula

$$P(\max(C(T_0, T_n) - \bar{K}, 0)) = \frac{P(\bar{K})}{\text{DV01}(\bar{K})} + \int_{\bar{K}}^\infty h(K) P(K) \mathrm{d}K, \quad (23)$$

where

$$h(K) = \left\{ (K - \bar{K}) \left[2 \frac{(\text{DV01}'(K))^2}{\text{DV01}^3(K)} - \frac{\text{DV01}''(K)}{\text{DV01}^2(K)} \right] - 2 \frac{\text{DV01}'(K)}{\text{DV01}^2(K)} \right\}.$$

Here $\text{DV01}(y)$ is again defined by (21).

Formula (23) is widely used by sophisticated practitioners to value CMS caps.

Example 3:
First consider a CMS rate $C(T_0, T_n)$ to be paid at some date $p \geq T_0$. We assume now a linear swap rate model. In this case, in view of (17) the distribution function μ is given by

$$\mu(y) = \alpha + 2\beta_p y,\ y \geq 0,$$

with α and β_p given by (14) and (15), respectively. The pricing equation is therefore

$$P(C(T_0, T_n) \text{ paid at time } p) = P(0)\alpha + \int_0^\infty P(K) 2\beta_p \mathrm{d}K. \quad (24)$$

For a CMS caplet $\max(C(T_0, T_n) - \bar{K}, 0)$ paid at time $p \geq T_0$ the distribution μ is

$$\mu(y) = \begin{cases} 0 & \text{if } y < \bar{K}, \\ (\alpha - \bar{K}\beta_p) + 2\beta_p y & \text{if } y \geq \bar{K}, \end{cases}$$

and we end up with the valuation formula

$$P(\max(C(T_0,T_n)-\bar{K},0) \text{ paid at time } p) = P(\bar{K})(\alpha+\beta_p\bar{K}) + \int_{\bar{K}}^{\infty} P(K)2\beta_p \mathrm{d}K. \quad (25)$$

6 Numerical examples

In this section we illustrate the smile effect on the convexity by some numerical examples. The examples are based on the EUR interest rate curve as of November 1, 2005. The market rates have been converted to continuously compounded zero rates on an act/365 basis:

1W	2.099%	2Y	2.750%
2W	2.109%	3Y	2.879%
1M	2.130%	4Y	2.988%
2M	2.225%	5Y	3.088%
3M	2.259%	6Y	3.171%
6M	2.364%	7Y	3.255%
9M	2.457%	8Y	3.337%
12M	2.534%	9Y	3.419%
18M	2.641%	10Y	3.486%
		12Y	3.605%
		15Y	3.741%
		20Y	3.879%

6.1 Libor-in-arrears swaps and in-arrears caps

We investigate Libor-in-arrears swaps and in-arrears caps as discussed in Example 1 above. The cap market quotes prices in terms of an implied flat[3] volatility for all strike levels. As is standard in practice from these flat volatilities one can extract the implied volatilities for each individual caplet and each strike level. Here is the caplet volatility surface as of November 1, 2005 which is the primary input for obtaining the price $P(K)$ of an individual caplet with strike K:

[3] This means that when pricing a cap, all caplets of the cap are priced using one and the same volatility in the market BLACK 76 valuation formula.

caplet start/strike	1.50%	1.75%	2.00%	2.25%	2.50%	3.00%
6M	32.00%	28.50%	23.20%	20.10%	19.00%	19.20%
1Y6M	31.14%	27.93%	24.65%	22.91%	22.25%	21.78%
2Y6M	31.25%	28.18%	25.08%	23.70%	23.11%	21.85%
3Y6M	29.40%	27.10%	24.17%	22.79%	22.25%	21.22%
4Y6M	28.83%	25.91%	24.12%	22.38%	22.02%	20.49%
5Y6M	27.39%	26.00%	23.83%	22.73%	22.04%	20.78%
6Y6M	27.58%	24.79%	23.14%	22.02%	21.39%	19.80%
7Y6M	25.95%	24.20%	22.17%	21.98%	21.34%	19.60%
8Y6M	24.61%	23.48%	21.79%	20.92%	20.22%	19.17%
9Y6M	25.32%	21.88%	21.19%	20.77%	20.09%	18.80%
10Y6M	23.71%	23.18%	21.18%	19.79%	20.19%	18.40%
11Y6M	23.06%	21.98%	20.51%	19.44%	19.74%	17.99%
12Y6M	23.74%	21.43%	20.47%	19.70%	18.46%	17.20%
13Y6M	22.64%	20.70%	19.84%	19.06%	18.21%	16.78%
14Y6M	22.09%	20.02%	19.39%	18.70%	17.63%	16.26%
15Y6M	22.60%	21.23%	19.85%	19.01%	18.53%	17.12%
16Y6M	21.86%	20.35%	19.29%	18.53%	17.85%	16.43%
17Y6M	21.42%	20.04%	18.98%	18.24%	17.61%	16.19%
18Y6M	20.89%	19.57%	18.60%	17.90%	17.24%	15.81%
19Y6M	20.24%	19.02%	18.14%	17.48%	16.83%	15.40%
caplet start/strike	3.50%	4.00%	5.00%	6.00%	7.00%	8.00%
6M	22.00%	23.90%	27.50%	31.30%	33.70%	34.60%
1Y6M	21.64%	21.16%	21.75%	24.27%	27.21%	28.51%
2Y6M	22.00%	22.52%	23.84%	24.79%	25.64%	26.52%
3Y6M	20.59%	20.84%	21.76%	23.17%	24.29%	25.14%
4Y6M	20.10%	20.11%	20.86%	21.59%	22.65%	23.39%
5Y6M	19.82%	19.67%	19.69%	20.76%	21.66%	22.54%
6Y6M	19.22%	18.40%	19.07%	19.85%	20.99%	21.49%
7Y6M	18.67%	18.25%	17.96%	18.69%	19.38%	20.26%
8Y6M	17.29%	17.36%	17.61%	18.15%	18.78%	19.07%
9Y6M	17.68%	16.66%	16.56%	16.97%	17.44%	18.29%
10Y6M	17.83%	17.31%	16.89%	17.02%	17.61%	17.91%
11Y6M	17.26%	16.48%	15.98%	16.04%	16.54%	16.95%
12Y6M	16.07%	15.90%	15.53%	15.98%	16.47%	17.11%
13Y6M	15.70%	15.36%	14.85%	15.14%	15.60%	16.20%
14Y6M	15.09%	14.83%	14.26%	14.58%	15.02%	15.65%
15Y6M	16.37%	15.59%	15.29%	15.47%	16.09%	16.60%
16Y6M	15.54%	14.89%	14.47%	14.68%	15.27%	15.84%
17Y6M	15.35%	14.61%	14.21%	14.39%	15.01%	15.58%
18Y6M	14.95%	14.20%	13.77%	13.94%	14.58%	15.16%
19Y6M	14.56%	13.77%	13.34%	13.49%	14.15%	14.76%

Now applying[4] the replication formulae (18), (19) to a Libor-in-arrears swap or an in-arrears cap the questions arises what would be a reasonable benchmark for comparing the results? We compare the results from our replication with the results of a somewhat naive but quite popular approach in practice. There, one uses just a convexity adjusted forward rate to take care of the in-arrears effect. The convexity adjusted forward rate is calculated as in (20) with σ set to the at-the-money volatility. For an in-arrears caplet then, in addition to the adjusted forward rate, the volatility for the caplet is taken from the smile according to the strike rate of the caplet. Of course, from the modeling point of view this seems to be a rather inconsistent way of taking care of the in-arrears and the smile effect simultaneously. In the table below the corresponding results are labeled “adj Fwd”. A second benchmark is based on a convexity adjusted forward as above, but the original volatility σ for the respective strike is also adjusted for the in-arrears effect to a new volatility σ^* by the formula

$$(\sigma^*)^2 = \sigma^2 + \ln\left[\frac{(1+\Delta L_0(T_1,T_2))(1+\Delta L_0(T_1,T_2)\exp(2\sigma^2 T_1))}{(1+\Delta L_0(T_1,T_2)\exp(\sigma^2 T_1))^2}\right]/T_1,$$

cf. [1], Section 4.1. This is labeled “adj Fwd & Vol”.

Here are the results for a 10 years in-arrears cap with semi-annual periods[5] for different strikes. The first row shows the sum of all caplets with individual caplet prices in the rows below. The case of strike $\bar{K} = 0\%$ corresponds to the floating leg of a Libor-in-arrears swap.

[4] We use Gauss Legendre quadrature and integrate over strikes from 0% to 20%.
[5] The caps consist of 19 caplets, with the first one starting in 6 months.

strike $\bar{K} = 0\%$			strike $\bar{K} = 2\%$		
Replication	adj Fwd	adj Fwd & Vol	Replication	adj Fwd	adj Fwd & Vol
29.072%	29.047%	29.047%	13.553%	13.528%	13.538%
1.414%	1.414%	1.414%	0.427%	0.427%	0.427%
1.476%	1.476%	1.476%	0.522%	0.522%	0.522%
1.527%	1.527%	1.527%	0.570%	0.570%	0.570%
1.473%	1.473%	1.473%	0.561%	0.561%	0.561%
1.532%	1.532%	1.532%	0.634%	0.634%	0.634%
1.508%	1.507%	1.507%	0.631%	0.630%	0.630%
1.570%	1.569%	1.569%	0.703%	0.702%	0.703%
1.527%	1.526%	1.526%	0.691%	0.690%	0.691%
1.569%	1.568%	1.568%	0.737%	0.735%	0.736%
1.517%	1.516%	1.516%	0.718%	0.717%	0.718%
1.588%	1.586%	1.586%	0.788%	0.786%	0.787%
1.542%	1.540%	1.540%	0.769%	0.767%	0.768%
1.612%	1.610%	1.610%	0.834%	0.832%	0.833%
1.523%	1.521%	1.521%	0.789%	0.788%	0.789%
1.609%	1.607%	1.607%	0.864%	0.862%	0.863%
1.551%	1.549%	1.549%	0.836%	0.834%	0.835%
1.524%	1.522%	1.522%	0.825%	0.823%	0.824%
1.494%	1.492%	1.492%	0.811%	0.809%	0.810%
1.513%	1.511%	1.511%	0.840%	0.838%	0.839%

strike $\bar{K} = 4\%$			strike $\bar{K} = 6\%$		
Replication	adj Fwd	adj Fwd & Vol	Replication	adj Fwd	adj Fwd & Vol
3.968%	3.939%	3.951%	1.420%	1.394%	1.405%
0.010%	0.010%	0.010%	0.001%	0.001%	0.001%
0.033%	0.033%	0.033%	0.002%	0.002%	0.002%
0.064%	0.064%	0.064%	0.009%	0.008%	0.008%
0.085%	0.085%	0.085%	0.016%	0.016%	0.016%
0.123%	0.122%	0.122%	0.029%	0.028%	0.029%
0.129%	0.129%	0.129%	0.034%	0.033%	0.034%
0.173%	0.171%	0.172%	0.050%	0.049%	0.049%
0.179%	0.177%	0.178%	0.053%	0.052%	0.052%
0.211%	0.209%	0.210%	0.067%	0.066%	0.066%
0.214%	0.212%	0.213%	0.071%	0.069%	0.070%
0.254%	0.252%	0.253%	0.093%	0.091%	0.091%
0.249%	0.247%	0.248%	0.094%	0.092%	0.093%
0.297%	0.295%	0.296%	0.114%	0.112%	0.113%
0.289%	0.288%	0.288%	0.109%	0.107%	0.108%
0.335%	0.333%	0.334%	0.135%	0.133%	0.134%
0.325%	0.323%	0.324%	0.135%	0.133%	0.134%
0.323%	0.321%	0.322%	0.133%	0.130%	0.131%
0.319%	0.317%	0.318%	0.128%	0.126%	0.127%
0.353%	0.351%	0.352%	0.148%	0.145%	0.147%

strike $\bar{K} = 8\%$			strike $\bar{K} = 10\%$		
Replication	adj Fwd	adj Fwd & Vol	Replication	adj Fwd	adj Fwd & Vol
0.694%	0.674%	0.684%	0.400%	0.386%	0.395%
0.000%	0.000%	0.000%	0.000%	0.000%	0.000%
0.001%	0.001%	0.001%	0.000%	0.000%	0.000%
0.002%	0.002%	0.002%	0.001%	0.001%	0.001%
0.004%	0.004%	0.004%	0.002%	0.002%	0.002%
0.010%	0.009%	0.009%	0.004%	0.004%	0.004%
0.013%	0.013%	0.013%	0.006%	0.006%	0.006%
0.020%	0.020%	0.020%	0.011%	0.010%	0.010%
0.022%	0.022%	0.022%	0.012%	0.012%	0.012%
0.030%	0.030%	0.030%	0.017%	0.016%	0.017%
0.034%	0.033%	0.033%	0.019%	0.019%	0.019%
0.046%	0.044%	0.045%	0.027%	0.026%	0.026%
0.047%	0.046%	0.047%	0.028%	0.027%	0.027%
0.059%	0.057%	0.058%	0.034%	0.033%	0.033%
0.056%	0.055%	0.056%	0.032%	0.030%	0.031%
0.069%	0.067%	0.068%	0.040%	0.039%	0.039%
0.066%	0.065%	0.066%	0.039%	0.038%	0.039%
0.068%	0.066%	0.067%	0.040%	0.039%	0.039%
0.068%	0.066%	0.067%	0.040%	0.039%	0.040%
0.078%	0.076%	0.077%	0.049%	0.047%	0.048%

Overall, the results are quite close and the two naive approaches (adjusted forward rates without and with additional adjusted volatilities) yield results whose differences to the correct one from replication are negligible in practice. In general the results from adjusted forwards and adjusted volatilities are somewhat closer to the correct ones.

These are good news for practitioners indicating that naive approaches are fairly sufficient for in-arrears caps and swaps.

6.2 Constant maturity swaps and CMS caps

In this section we investigate CMS swaps and CMS caps as discussed in Examples 2 and 3 above. The swaption market quotes at-the-money implied volatilities for a variety of swaption maturities and tenors for the underlying swap. Our numerical examples are based on the following at-the-money swaption volatilities as of November 1, 2005.

opt / swap	1Y	2Y	3Y	4Y	5Y
1M	18.50%	21.80%	21.60%	21.10%	20.40%
2M	19.60%	22.00%	21.70%	21.20%	20.50%
3M	20.80%	22.00%	21.70%	21.10%	20.40%
6M	20.90%	21.50%	21.10%	20.40%	19.90%
9M	22.00%	21.30%	20.70%	20.10%	19.50%
1Y	22.20%	21.50%	20.70%	20.00%	19.30%
18M	21.80%	21.00%	20.20%	19.50%	18.90%
2Y	21.60%	20.70%	19.80%	19.20%	18.60%
3Y	20.80%	20.00%	19.20%	18.50%	18.00%
4Y	20.00%	19.30%	18.50%	17.80%	17.30%
5Y	19.20%	18.40%	17.70%	17.20%	16.80%
7Y	17.60%	16.80%	16.20%	15.90%	15.70%
10Y	16.10%	15.40%	15.00%	14.80%	14.70%

opt / swap	6Y	7Y	8Y	9Y	10Y
1M	19.70%	18.90%	18.20%	17.60%	17.00%
2M	19.80%	19.00%	18.30%	17.70%	17.10%
3M	19.70%	18.90%	18.20%	17.60%	17.10%
6M	19.20%	18.50%	18.00%	17.50%	17.00%
9M	18.90%	18.30%	17.80%	17.30%	16.90%
1Y	18.70%	18.20%	17.70%	17.30%	16.90%
18M	18.40%	17.90%	17.40%	17.00%	16.70%
2Y	18.10%	17.60%	17.20%	16.90%	16.60%
3Y	17.50%	17.10%	16.80%	16.50%	16.30%
4Y	17.00%	16.70%	16.40%	16.20%	16.10%
5Y	16.50%	16.30%	16.10%	15.90%	15.80%
7Y	15.60%	15.40%	15.30%	15.20%	15.10%
10Y	14.60%	14.50%	14.50%	14.50%	14.50%

The swaption smile is rarely quoted on publicly available data sources. Our analysis below is based on a swaption smile surface that is constructed by adding certain volatility shifts to the quoted at the money volatility. The shift to apply depends on the respective strike offset relative to the corresponding at-the-money strike rate. The smile table below refers to options to enter into a 10Y swap.

opt/offset	-100.00%	-50.00%	-30.00%	30.00%	50.00%	100.00%	125.00%
1M	45.61%	6.74%	2.05%	0.06%	0.16%	1.85%	6.99%
3M	45.88%	6.78%	2.06%	0.06%	0.16%	1.86%	7.03%
6M	45.61%	6.74%	2.05%	0.06%	0.16%	1.85%	6.99%
12M	45.34%	6.71%	2.04%	0.06%	0.16%	1.83%	6.95%
2Y	44.53%	6.59%	2.00%	0.06%	0.16%	1.80%	6.83%
3Y	43.73%	6.47%	1.96%	0.06%	0.16%	1.77%	6.70%
5Y	42.39%	6.27%	1.90%	0.05%	0.15%	1.72%	6.50%
10Y	38.90%	5.75%	1.75%	0.05%	0.14%	1.57%	5.96%
20Y	35.41%	5.24%	1.59%	0.05%	0.13%	1.43%	5.43%

To give an example on how to understand the smile table consider a 2Y into 10Y swaption struck at 150% of the at-the money rate. The volatility for this swaption is then the 2Y into 10Y at-the-money volatility of 16.60% plus 0.16%.

Now, applying[6] the replication formulae (23), (25) to a CMS cap we compare the results again to the results from naive approaches based on convexity adjusted forward CMS rates and adjusted volatilities. Of course, the volatility for the cap strike is always taken by interpolation from the swaption volatility surface. The convexity adjusted forward swap rate is calculated from the forward swap rate $C_0(T_0, T_n)$ (cf. (16)) by

$$C_0(T_0, T_n)\left[1 + \left(1 - \frac{B(0,T_0) - B(0,T_n)}{C_0(T_0,T_n)B(0,p)\sum_{i=1}^{n}\Delta_i}\right)(\exp(\sigma_{\mathrm{atm}}^2 T_0) - 1)\right]$$

using the at-the-money volatility σ_{atm}, cf. [1], formula (28). Below the results of this approach are labeled "adj Fwd". Results labeled as "adj Fwd & Vol" refer to the situation where, in addition to the adjustment of the forward rate, also the volatility σ for the respective strike is adjusted for the CMS effect to a new volatility σ^* calculated by (see [1], Section 4.1)

$$(\sigma^*)^2 = \sigma^2 + \ln\left[\frac{(\alpha + \beta_p C_0(T_0,T_n))(\alpha + \beta_p C_0(T_0,T_n)\exp(2\sigma^2 T_0))}{(\alpha + \beta_p C_0(T_0,T_n)\exp(\sigma^2 T_0))^2}\right]/T_0,$$

where α, β_p are defined in (14), (15).

Here are the results for a 10 years CMS cap with semi-annual periods for different strikes. The tenor of the CMS rate is 10Y. The first row shows the sum of all caplets with individual caplet prices in the rows below. The case of strike $\bar{K} = 0\%$ corresponds to the floating leg of a CMS swap.

[6] Again we use Gauss Legendre quadrature.

strike $\bar{K} = 0\%$			
Replication (25)	Replication (23)	adj Fwd	adj Fwd & Vol
33.791%	33.835%	33.561%	33.561%
1.797%	1.790%	1.797%	1.797%
1.785%	1.779%	1.785%	1.785%
1.846%	1.842%	1.845%	1.845%
1.812%	1.808%	1.811%	1.811%
1.817%	1.814%	1.815%	1.815%
1.821%	1.818%	1.818%	1.818%
1.830%	1.829%	1.826%	1.826%
1.809%	1.809%	1.803%	1.803%
1.833%	1.834%	1.825%	1.825%
1.797%	1.799%	1.787%	1.787%
1.817%	1.820%	1.805%	1.805%
1.787%	1.792%	1.773%	1.773%
1.810%	1.815%	1.793%	1.793%
1.728%	1.734%	1.709%	1.709%
1.765%	1.773%	1.744%	1.744%
1.717%	1.726%	1.693%	1.693%
1.694%	1.704%	1.668%	1.668%
1.670%	1.681%	1.642%	1.642%
1.656%	1.668%	1.625%	1.625%

strike $\bar{K} = 2\%$			
Replication (25)	Replication (23)	adj Fwd	adj Fwd & Vol
17.878%	17.922%	17.679%	17.719%
0.800%	0.793%	0.800%	0.800%
0.819%	0.813%	0.819%	0.819%
0.869%	0.865%	0.869%	0.869%
0.873%	0.869%	0.872%	0.872%
0.894%	0.891%	0.893%	0.893%
0.916%	0.913%	0.913%	0.913%
0.938%	0.937%	0.934%	0.935%
0.944%	0.944%	0.939%	0.940%
0.972%	0.973%	0.965%	0.966%
0.968%	0.969%	0.959%	0.961%
0.992%	0.994%	0.981%	0.983%
0.988%	0.992%	0.976%	0.978%
1.011%	1.016%	0.997%	1.000%
0.975%	0.982%	0.959%	0.962%
1.006%	1.013%	0.987%	0.991%
0.987%	0.996%	0.966%	0.970%
0.981%	0.991%	0.958%	0.962%
0.974%	0.984%	0.949%	0.954%
0.972%	0.984%	0.945%	0.950%

strike $\bar{K} = 4\%$			
Replication (25)	Replication (23)	adj Fwd	adj Fwd & Vol
5.161%	5.189%	4.956%	5.000%
0.023%	0.015%	0.023%	0.023%
0.065%	0.057%	0.064%	0.065%
0.105%	0.100%	0.105%	0.105%
0.139%	0.134%	0.138%	0.139%
0.172%	0.167%	0.170%	0.171%
0.203%	0.200%	0.201%	0.202%
0.233%	0.231%	0.229%	0.231%
0.258%	0.257%	0.253%	0.255%
0.286%	0.286%	0.279%	0.281%
0.303%	0.304%	0.294%	0.297%
0.327%	0.329%	0.316%	0.319%
0.342%	0.345%	0.329%	0.332%
0.363%	0.367%	0.348%	0.351%
0.362%	0.368%	0.345%	0.349%
0.385%	0.392%	0.366%	0.369%
0.389%	0.397%	0.368%	0.371%
0.395%	0.406%	0.372%	0.376%
0.401%	0.411%	0.376%	0.380%
0.408%	0.420%	0.381%	0.385%

strike $\bar{K} = 6\%$			
Replication (25)	Replication (23)	adj Fwd	adj Fwd & Vol
1.415%	1.449%	1.213%	1.251%
0.000%	0.000%	0.000%	0.000%
0.000%	0.000%	0.000%	0.000%
0.002%	0.002%	0.002%	0.002%
0.007%	0.007%	0.006%	0.006%
0.014%	0.014%	0.013%	0.013%
0.023%	0.018%	0.021%	0.022%
0.035%	0.030%	0.032%	0.032%
0.047%	0.044%	0.042%	0.044%
0.061%	0.059%	0.054%	0.055%
0.073%	0.072%	0.064%	0.066%
0.086%	0.086%	0.076%	0.078%
0.098%	0.100%	0.085%	0.088%
0.111%	0.114%	0.096%	0.099%
0.117%	0.122%	0.101%	0.104%
0.132%	0.138%	0.112%	0.116%
0.140%	0.147%	0.118%	0.122%
0.148%	0.158%	0.124%	0.129%
0.156%	0.165%	0.130%	0.135%
0.165%	0.175%	0.136%	0.141%

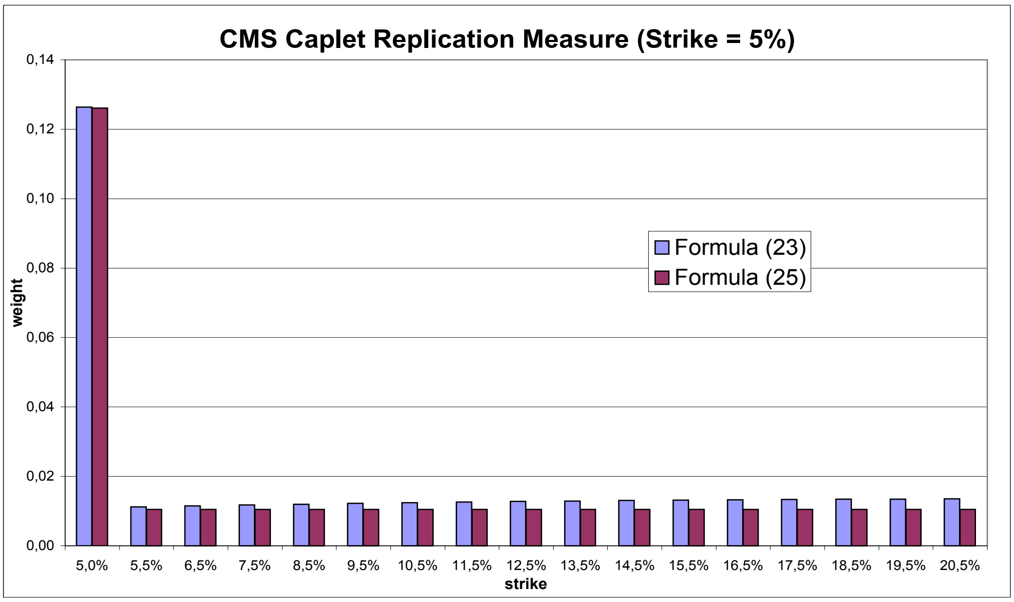
For all strikes $\bar{K}$ we see significant differences between the prices obtained from replication compared to the prices from naive adjustment approaches. In case of a CMS swap (i.e., $\bar{K} = 0\%$), the effect from the swaption smile

leads to a difference of about 2.8 basis points in case of replication (25) or 3.4 basis points for a replication based on (23) in the fair fixed rate relative to the naive valuation. Clearly, this cannot be ignored[7].

The naive approaches, although taking into account the smile by taking the caplet volatility from the swaption smile, show a significant mispricing relative to the correct valuations based on the replication (23), (25). Observe that the results are sensitive to the significance of the swaption smile.

Comparing the results of the two replication approaches (23), (25), it turns out that the replication based on the idea of cash-settled swaptions consistently leads to slightly higher CMS caplet prices. It is not clear which approach is superior, since both rely on assumptions that are hard to compare and value against each other.

The following figure shows the weights that the signed measure μ assigns to swaptions with different strikes during the replication of a CMS caplet[8] according to the two alternative replications (23), (25).



[7]There is a recent discussion between market participants about differences in the valuation of CMS swaps between the results of standard (naive) convexity adjustment valuations and quotes by brokers. It seems that these differences can be attributed to the swaption smile.

[8]This is the caplet starting in 9.5 years on a 10Y CMS rate. The replication integral was discretized in 1% steps.

7 Conclusions

We have shown how to evaluate quite popular exotic interest rate derivatives such as Libor-in-arrears caps or CMS caps incorporating the volatility smile present in the cap and swaption market. It turns out that the volatility smile has a significant impact, which, in particular, cannot be ignored when pricing CMS swaps or caps. For Libor-in-arrears derivatives, the smile effect is already relatively accurately captured by naive approaches based on convexity adjustments.

References

[1] Boenkost, W., Schmidt, W.: Notes on convexity and quanto adjustments for interest rates and related options, Research Report No. 47, HfB - Business School of Finance & Management, 2003 http://www.hfb.de/Dateien/Arbeits47e.pdf

[2] Breeden, D., Litzenberger, R.: Prices of state-contingent claims implicit in option prices, *J. Business*, **51**, 621-651, 1978

[3] Brigo, D., Mercurio, F.: *Interest Rate Models - Theory and Practice*, Springer, 2001

[4] Hagan, P.S.: Convexity Conundrums: Pricing CMS Swaps, Caps and Floors, *Wilmott Magazine*, March 2003, pp.38-44

[5] Hunt, P.J., Kennedy, J.E.: *Financial Derivatives in Theory and Practice*, Wiley, 2000

[6] Pelsser, A.: *Efficient Models for Valuing Interest Rate Derivatives*, Springer, 2000